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NON-LINEAR DEVELOPMENT OF ROTATING STALL IN AN AXIAL COMPRESSOR AT A NEAR-CRITICAL FLOW RATE†

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The development of rotating stall (RS) at a flow rate close to the critical value at which rotating stall occurs is considered using a modified Moore–Greitzer model. Landau's equation is derived for the amplitude of the first mode. The theory is used to describe the experimental laboratory compressor used in the Massachusetts Institute of Technology to investigate rotating stall. It is established that this compressor is atypical in the sense that the nature of the excitation of rotating stall in it is, in a certain sense, intermediate between subcritical and supercritical, or, more precisely, an insignificant change in the compressor parameters, within the limits of experimental error, can lead to a change in the type of excitation.

Rotating stall (RS) is a special form of hydrodynamic instability of the flow in a compressor. The development of this instability leads to the onset of a flow regime in which the flow around a group of rotor and (or) stator blades becomes separated whereupon, as time passes, the separation on the blades on one side of this group disappears while it appears on the other side of this group so that, as a whole, the domain with a separated flow around the blades rotates around the compressor axis [1].

As a result of the advent of actively stabilized compressors [2] it has been found that, in spite of the high complexity of the flow (a complex compressor geometry which varies with time, an extremely unsteady, on the average, and turbulent nature of the flow as a whole and the occurrence of separation), the onset of RS in many cases can be adequately described using comparatively simple models which, in essence, are based on taking the limit as the number of blades tends to infinity (a review of such models is given in [3]). These models make use of empirical information on the characteristics of a compressor when there is no RS in order to predict when it will occur. The necessary empirical information has been obtained by direct measurements on a compressor, RS in which was prevented by means of an active control system, a linear stability has been constructed and the critical flow rate has been calculated at which RS sets in and the phase velocities and the increments in the growth of several modes have been found [2].

If the flow rate is exactly equal to the critical flow rate, then, according to the linear theory, the amplitude of a neutral perturbation will remain constant. Actually, relatively weak non-linear effects will lead to a slow change in the amplitude and must therefore be taken into account. This also holds for values of the flow rate which are only slightly different from the critical value when the rate of change in the amplitude of the perturbations, calculated using linear theory, turns out to be of the same order as its rate of change due to weak non-linear effects. A study of the behaviour of a system under the conditions which have been described is the subject of the so-called weakly-non-linear stability theory. It is found that, under fairly general assumptions, the equations describing the change in amplitude of a perturbation has a universal form regardless of the nature of the physical system considered. This first-order ordinary differential equation with a cubic non-linearity on the right-hand side, which is called Landau's equation [4, §26], contains just two coefficients which depend on the parameters of the actual physical system. The main result of the calculation of these coefficients is found to be the possibility of determining whether the stability loss is subcritical or supercritical, that is, to understand whether the amplitude under steady-state conditions is small or at once finite at values of the flow rate only slightly less than the critical flow rate (flows with flow rate greater than the critical flow rate are stable) and, correspondingly, whether there is hysteresis of the RS when there are changes in the flow rate.

The aim of this paper is to obtain general formulae for the coefficients of Landau's equation within the framework of the modified Moore–Greitzer model [2, 3] and to apply these formulae to the case of an actual laboratory compressor.

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1. FORMULATION OF THE PROBLEM

Consider the flow of an incompressible fluid in a compressor with short blades, the length of which is much smaller than the radius of the rotors and stators, with a cylindrical hub and housing of slightly differing diameters. In this case, the dependence of the parameters on the distance from the rotation axis can be neglected and a two-dimensional flow is considered which is solely dependent on the axial and circumferential coordinates.

We shall use the modified Moore–Greitzer model which consists of the following [2, 3]. The flow domain is subdivided into three parts: an inlet channel, a domain which is occupied by the fixed and moving blades and an outlet channel.

In the inlet and outlet channels, the flow is assumed to be inviscid and is described by the Euler equations. The flow in the region of the blades is described using a semi-empirical model which associates the pressure drop between the outlet and the inlet in the region of the blades with the axial component of the velocity

$$p_{\text{out}} - p_{\text{in}}^{*} = \psi(\varphi) - s - r - \nu\frac{\partial\varphi}{\partial\theta} - \mu\frac{\partial\varphi}{\partial t} \tag{1.1}$$

$$\sigma(\varphi)\frac{\partial s}{\partial t} = S(\varphi) - s \tag{1.2}$$

$$\tau(\varphi)\left(\frac{\partial r}{\partial t} + \frac{\partial r}{\partial\theta}\right) = R(\varphi) - r \tag{1.3}$$

Here, $p_{\text{out}}(\theta, t)$ is the static pressure at the outlet from the blade domain, $p_{\text{in}}^{*}(\theta, t)$ is the stagnation pressure at the inlet to the blade domain, $\varphi(\theta, t)$ is the axial velocity at the inlet and the outlet from the blade domain (these velocities are the same by virtue of the mass conservation law), $\psi(\varphi)$ is the idealized compressor characteristic, the difference between the static pressure at the outlet and the total pressure at the inlet under steady conditions when there is no losses and RS, s and r are the losses in total pressure in the stators and rotors, ν and μ are constant parameters which characterize the inertia of the fluid in the rotors and in the entire domain occupied by the blades respectively, σ and τ are the characteristic time lags of the losses in the stators and the rotors respectively, t is the time and θ is the circumferential angle. All quantities are reduced to dimensionless form using the assumption that the mean radius of the flow domain, the circumferential velocity of the rotor blades and the density of the fluid are equal to unity, without any loss of generality.

The functions $\psi(\varphi)$, $S(\varphi)$ and $R(\varphi)$ are characteristics of a compressor operating under steady conditions. In the subsequent analysis, they are assumed to be specified as well as the values of the constants ν and μ. Furthermore, it is assumed that $\sigma(\varphi)$ and $\tau(\varphi)$ are directly proportional to the time needed for a fluid particle to pass through an interblade channel, i.e. $\sigma(\varphi) = k_\sigma/\varphi$ and $\tau(\varphi) = k_\sigma/\varphi$, where the constants $k_\sigma > 0$ and $k_\tau > 0$ (generally speaking, different) are also assumed given.

The physical meaning of this empirical model is simple. Equation (1.2) describes the delay of the losses in the stators when the axial velocity is varied. The delay of the losses in the rotors is governed by Eq. (1.3) in which the derivatives with respect to θ appear as a result of the transition from a frame of reference which rotates together with the rotors to a fixed frame of reference. For the same reason, derivatives with respect to θ also appear in Eq. (1.1) in which the terms with derivatives with respect to θ and t describe the pressure drop which occurs as a result of the acceleration of the fluid when the axial velocity is varied.

The flows in the three domains are matched on the boundaries between them using continuity conditions for the pressure and the axial velocity. Moreover, at the outlet from the region containing the blades, the circumferential component of the velocity becomes equal to zero as a result of the high-solidity last stator. This condition is also specified as a boundary condition for the flow in the outlet channel.

For simplicity, we will assume that the inlet and outlet channels extend to infinity upstream and downstream. The constant axial and circumferential components of the velocity (the circumferential velocity is equal to zero) are specified far upstream and it is sufficient to require that all the characteristics approach constant values far downstream.

Naturally, all the parameters must be periodic functions of θ with a period of 2π.

The Euler equations with the above-mentioned boundary conditions at infinity and matching conditions and relations (1.1)–(1.3) constitute a closed mathematical model within the framework of which all the subsequent analysis is carried out. Note that relations (1.1)–(1.3) and the condition of a zero circumferential component of the velocity at the outlet from the blade region can be considered as specific conditions at a certain singularity in the flow of an ideal fluid.

The system under consideration has a steady solution in which $\varphi = \varphi_s = \text{const}$. By virtue of the periodicity with respect to θ, the unsteady solution can be represented in the form of a Fourier series

$$\varphi = \varphi_s + \sum_{n\neq 0} A_n(t)e^{in\theta}, \quad A_n(t) = \bar{A}_{-n}(t) \tag{1.4}$$

In the linear approximation all $A_n(t) = C_n \exp \lambda_n t$. In the case of the actual characteristics of a compressor, the real part of all the λ_n in (1.1)–(1.3) is negative for φ_s greater than a certain critical value φ_c. As φ_s is decreased, first the mode with $n = 1$ at $\varphi_s = \varphi_c$ and, subsequently, the remaining modes lose stability in order of increasing n [2, 3].

The properties which have been enumerated signify that the system under consideration satisfies all the requirements for the applicability of the general Landau theory ([4], §26). Hence, in the case of flow rate which are close to the critical flow rate and small amplitudes of the perturbations, the amplitude of the first mode satisfies Landau's equation

$$d|A_1|/dt = \operatorname{Re}\lambda_1 |A_1| + K|A_1|^3 \tag{1.5}$$

The problem consists of deriving this equation by an asymptotic analysis when $\varphi_s \to \varphi_c, A_n \to 0$ and applying the result obtained to an actual compressor in order to determine the nature of the excitation of RS in it.

We emphasize that the case of a fixed flow rate through the compressor at a constant velocity of motion of the rotor blades is considered. The interaction of the compressor with the other parts of the engine or hydraulic circuit in which a compressor operates can lead to other types of oscillational and hysteresis phenomena, such as, for example, surge and surge hysteresis.

2. SMALL-AMPLITUDE PERTURBATIONS

In accordance with the formulation of the problem which has been adopted, the flow in the input channel is described by the Euler equations. Moreover, since homogeneous boundary conditions are specified far upstream, that is, the vorticity is equal to zero, this is a potential flow. At the inlet to the blades, the axial velocity of the flow, by the matching condition, is equal to $\varphi(t, \theta)$. With these boundary conditions, it is easy to solve the potential flow equations by the method of separation of variables and, using the unsteady Bernoulli theorem, to express [2, 3] the stagnation pressure at the inlet to the blades in terms of the coefficients of the expansion of φ in Fourier series (1.4) ($A'(t) = dA/dt$

$$p^*_{\text{in}} = p^*_{-\infty} - \sum_{n\neq 0} A'_n(t)e^{in\theta}/|n| \tag{2.1}$$

This formula holds for any amplitude of the perturbations.

The flow in the outlet channel is also described by the Euler equations but this flow is not potential. However, since both the magnitude and direction of the velocity at the outlet from the blades are specified, as is required for the unique determination of the solution of the Euler equations, the pressure at the outlet from the blades can also be expressed in terms of the Fourier coefficients of the function φ. The appropriate formula has been obtained in the linear approximation by many authors (see [3]), and up to terms of the third order in the amplitude in [5]. More precisely, this formula has an accuracy of the order of ε^3, if the amplitude of the perturbations is of the order of ε and the characteristic time of its variation is of the order of ε^{-2}. In fact, such an order for the characteristic time follows from (1.5) when $\varphi_s - \varphi_c \sim \varepsilon^2$, that is, when $\operatorname{Re}\lambda_1 \sim \varepsilon^2$. After the corresponding renormalization, this formula takes the form

$$p_{\text{out}} - p_{+\infty} = \sum_{n\neq 0}\left(A'_n/|n| + \sum_{k+l=n} \pi_{kl}(V/\varphi_s)A_k A_l + \right.$$

$$\left. + \sum_{j+k+l=n} \pi_{jkl}(V/\varphi_s)A_j A_k A_l/\varphi_s + \ldots\right)e^{in\theta} \tag{2.2}$$

Here V is the dimensionless phase velocity of the propagation of a neutral perturbation according to the linear theory. Formulae for calculating π_{kl} and π_{jkl} and plots of some of these coefficients, for the case zero circumferential velocity at the outlet from the last stator ($k_1 = 0$ in the notation previously adopted [5]) being considered in this paper, have been presented in [5].

Formulae (1.1)–(1.3), (2.1) and (2.2) form a system which is sufficient both for investigating linear stability as well as for deriving Eq. (1.5), which describes the non-linear behaviour of the amplitude at near-critical flow rate. The linear stability of the problem under consideration has been studied previously [2].

Representing the Fourier coefficients in the form $A_n = \varepsilon e^{\lambda_n t}$, $\varepsilon \ll 1$ and retaining in (1.1)–(1.3), (2.1) and (2.2) terms which are linear in ε, we obtain a cubic characteristic equation in λ_n

$$D_n(\lambda_n, \varphi_s) = -\left(\frac{2}{|n|} + \mu\right)\lambda_n + \psi'(\varphi_s) - in\nu - \frac{S'(\varphi_s)}{1+\sigma(\varphi_s)\lambda_n} - \frac{R'(\varphi_s)}{1+\tau(\varphi_s)(\lambda_n + in)} = 0 \tag{2.3}$$

In the case of neutral perturbations $\lambda_n = -inV$, where V is the phase velocity of the neutral wave. Under the conditions $\mu > \nu > 0$, $S' < 0$, $R' < 0$ which are usually satisfied, it follows from (2.3) that $0 < V < 1$. In the case of realistic model parameters, two roots of (2.3) always have a negative real part and a critical value of the flow rate φ_c exists at which the real part of the third root vanishes and the steady solution is unstable at flow rates less than the critical flow rate.

3. WEAKLY NON-LINEAR STABILITY THEORY

We will now consider the case when, as φs is reduced, the mode with $n = 1$ first loses stability. We change to a frame of reference which moves at the phase velocity V of the neutral wave, putting

$$y = \theta - Vt, \quad \varphi = \varphi_s + \sum_{n \neq 0} B_n(t) e^{iny}$$

Here, $B_n = A_n e^{inVt}$ and, since $dA_1/dt = \lambda_1(\varphi_s)A_l$ and $\lambda_1(\varphi_c) = -iV$, we have

$$dB_1/dt = (\lambda_1(\varphi_s) - \lambda_1(\varphi_c))B_1 + \ldots \tag{3.1}$$

Now, suppose that not only the amplitude of the perturbation is of the order of $\varepsilon \ll 1$ but the value of the flow rate is also so close to the critical value that $\varphi_s - \varphi_c \sim \varepsilon^2$. The right-hand side of (3.1) will then be of the order of ε^3. Consequently, in the case being considered, in order for the evolution of the amplitude of the perturbation to be described correctly, it is necessary to retain not only the linear terms but also the quadratic and cubic terms. In this case, the coefficients B_n with $n \neq 1$ occur in the equation for B_1. Although it follows from general considerations [4] that the quadratic terms do not appear in Landau's equation (1.5), it is necessary to take account of these terms in the intermediate calculations together with terms of the third order. The intermediate calculations are extremely laborious and are of no interest in their own right. They are therefore omitted here. Their general scheme is as follows: s and r in (1.2)–(1.3) are represented by their Fourier series and the coefficients of these series are then expressed in terms of B_n, while retaining terms up to the third order inclusively. The resulting expressions are then substituted into (1.1), terms with the like powers of $e^{i\theta}$ are collected and the leading terms in ε are retained. The final result for $n = 1$ and $n = 2$ can be written as follows:

$$\begin{gathered}\left(2+\mu-\frac{\tau R'}{(1+i(1-V)\tau)^2}-\frac{\sigma S'}{(1-iV\sigma)^2}\right)\frac{dB_1}{dt} = D'_{1\varphi}(-iV,\varphi_c)(\varphi_s-\varphi_c)B_1 + \\ +(\psi'''/2-\Pi_3-W_1)B_{-1}B_1^2+(\psi''-\Pi_2-W_2)B_{-1}B_2 \\ B_2=-B_1^2(\psi''/2-\pi_{1,1}-W_3)/D_2(-2iV,\varphi_c)\end{gathered} \tag{3.2}$$

Here

$$\Pi_2 = \pi_{-1,2} + \pi_{2,-1}, \quad \Pi_3 = \pi_{-1,1,1} + \pi_{1,-1,1} + \pi_{1,1,-1}$$

$$W_k = L_k[V\sigma, S] + L_k[(V-1)\tau, R], \quad k = 1,2,3$$

$$L_1[\gamma, Q(\varphi)] = \frac{1}{1-i\gamma}\left[\frac{Q'''(\varphi)}{2} - \frac{i\gamma Q''(\varphi)}{(1-2i\gamma)\varphi} - \left(\frac{2\gamma^2}{1+\gamma^2} - \frac{i\gamma}{(1-i\gamma)(1-2i\gamma)}\right)\frac{Q'(\varphi)}{\varphi^2}\right]$$

$$L_2[\gamma, Q(\varphi)] = \frac{1}{1-i\gamma}\left[Q''(\varphi) + \left(\frac{i\gamma}{1+i\gamma} - \frac{2i\gamma}{1-2i\gamma}\right)\frac{Q'(\varphi)}{\varphi}\right]$$

$$L_3[\gamma, Q(\varphi)] = \frac{1}{1-2i\gamma}\left[\frac{Q''(\varphi)}{2} - \frac{i\gamma Q'(\varphi)}{(1-i\gamma)\varphi}\right]$$

and τ, σ, R, S, ψ and their derivatives are calculated for $\varphi = \varphi_c$. We recall that τ and σ have the form const/φ.

Eliminating B_2 from these equations, we obtain an equation in B_1 which has the form

$$dB_1/dt = a(\varphi_s - \varphi_c)B_1 + bB_1^2 B_{-1}$$

The coefficients a and b are easily calculated from the preceding formula and are not given here for brevity. Of course, as would be expected from the form of (3.1), $a = d\lambda_1/d\varphi$ when $\varphi = \varphi_c$. The principal mode actually has the form $B_1 e^{iy} + B_{-1}e^{-iy}$, and $B_{-1} = \bar{B}_1$. Suppose that $B_1 = |B_1|e^{iY}$, that is, $|B_1|$ and Y are the real-valued amplitude and phase. Then,

$$\begin{aligned} d|B_1|/dt &= (\operatorname{Re} a)(\varphi_s - \varphi_c)|B_1| + (\operatorname{Re} b)|B_1|^3 \\ dY/dt &= (\operatorname{Im} a)(\varphi_s - \varphi_c) + (\operatorname{Im} b)|B_1|^2 \end{aligned} \tag{3.3}$$

The first of these equations is Landau's equation (1.5), since $|B_1| = |A_1|$. It follows from the second that, in a non-rotating frame of reference, points of stationary phase move at a velocity

$$V - dY/dt = V - (\operatorname{Im} a)(\varphi_s - \varphi_c) - (\operatorname{Im} b)|B_1|^2 \tag{3.4}$$

Since $\operatorname{Re} a = \operatorname{Re} d\lambda_1/d\varphi < 0$ always in actual compressors, i.e. instability sets in when the flow rate is reduced, then, as follows from Eq. (3.3) which has been obtained, the excitation of RS is subcritical when $\operatorname{Re} b > 0$ and supercritical when $\operatorname{Re} b < 0$.

4. NUMERICAL CALCULATIONS AND VERIFICATION OF THE CORRECTNESS OF THE FORMULAE OBTAINED

Since the intermediate calculations turned out to be extremely laborious, an independent check of the result is required. This check was carried out using the following methods. Calculations carried out by hand were then repeated using a system of computerized analytic calculations. The results were identical. Numerical calculations were then carried out in a frame of reference t, θ in order to avoid, as far as possible, the parallelism with the analytic calculations carried out in the t, y system. The unknown functions s, r and φ were represented by their expansion in Fourier series ((1.4) and, similarly, for r and s) in which only terms with $|n| < 3$ were retained. Actually, as is clear from (3.2), in order to check the results of weakly non-linear theory it is sufficient just to take account of these modes. However, in the interpretation of the numerical results, it is, of course, necessary to bear in mind that, in the case of a large perturbation amplitude, it is also necessary to take account of modes with higher numbers. Moreover, with the aim of additional simplification in the numerical calculations, no account was taken of the non-linearity of the response of the outlet channel of the compressor to a change in the axial velocity, i.e. the coefficients π_{kl} and π_{jkl} in (2.2) were equated to zero. However, as it actually turned out in the case of the actual example considered below, taking account of non-linearity in the outlet channel did not lead to any appreciable change in the result.

It is also necessary to point out that, whereas φ_s (the zeroth Fourier coefficient for the function φ) is constant, the zeroth coefficients for r and s depend on time, which was taken into account in both the analytic and the numerical calculations.

The results of the numerical calculations as well as the analytical formulae can be represented in the form of a dependence of the amplitude $|A_1| = |B_1|$ under steady conditions with respect to the amplitude on the mean axial velocity φ_s. The analytical result gives this dependence if one puts $d|B_1|/dt = 0$ in (3.3). As a result, a straight line $|B_1| = 0$ and a parabola $\varphi_s = \varphi_c - (\operatorname{Re} b/\operatorname{Re} a)|B_1|^2$ are obtained. In the case when a certain state with a steady amplitude is stable, it can be obtained by direct numerical modelling. In order to do this, the system of ordinary differential equations, obtained from

(1.1)–(1.3) by substitution of the Fourier series for the unknown functions and neglecting the coefficients with $|n| > 2$, was solved numerically in the case of a specified φ_s.

In the case when a state which is steady with respect to its amplitude is unstable, it cannot be calculated using the direct method described. Instead of this, the calculations were organized in the following manner. The calculation can be carried out up to a certain instant of time $t = T$ by specifying the amplitude of all the modes at the initial instant of time. The amplitude $|A_1|_{t=T}$ of the first mode at the instant T will be the function φ_s and the initial conditions including also the amplitude $|A_1|_{t=0}$. We must have $\Delta|A_1| = |A_1|_{t=T} - |A_1|_{t=0} = 0$ in the required steady state. Solving the equation $\Delta|A_1|(\varphi_s) = 0$ numerically (by the method of chords, for example) for the unknown φ_s, it is possible to find φ_s corresponding to a given $|A_1|$. The approach described is of an approximate nature, since the results of the calculation depend on the initial conditions for the other modes and for the phase of the first mode. Note, however, that, under conditions when the weakly non-linear stability theory is applicable, modes with $|n| = 1$ are approximately related by the corresponding neutral linear solution, i.e. the solution of the linearized problem when $\varphi_s = \varphi_c$, and all of the remaining modes have a much smaller magnitude. It is found that, if one specifies the initial conditions taking account of what has been said, then the results of the calculations depend very slightly on the existing arbitrariness in the choice of the initial conditions ($T = 50$ was taken in the calculations).

We emphasize that all the results and conclusions in this paper follow from the analytic solution obtained. The numerical calculations served to monitor the correctness of the result obtained analytically and as a source of illustrative material (see the figures below). The approximate nature of the results of the calculation of unstable steady-state conditions are not of any great significance. For this reason, a more detailed description of the numerical calculations is not presented here.

It turned out that the results obtained by the various methods: analytically by hand and using a computer system doing analytical calculations, numerically using the direct method and numerically using the approximate method for specified amplitudes, were found to be in complete agreement (as can be seen from the graphs presented in Section 6).

5. MODELLING OF THE MIT COMPRESSOR

In order to apply the results which have been obtained to an actual compressor, it is necessary to know the inertial parameters μ and ν, the time lag parameters $k_\tau = \tau\varphi$ and $k_\sigma = \sigma\varphi$ and the quasisteady-state characteristics $\psi(\varphi)$, $R(\varphi)$ and $S(\varphi)$. The characteristics which were subsequently used were obtained from data presented in [2] in the following way.

The unmodified Moore–Greitzer model (this model is (1.1)–(1.3) when $\sigma = \tau = 0$) gives the following expression for the frequency of the unperturbed motion: $nV = n\nu/(2/n + \mu) = \omega r/U$ in the notation of [2]. The values of nV for $n = 1, 2, 3$ are given in Fig. 19 of [2]. Direct measurements of the plot with a subsequent least-squares treatment give $\nu = 0.67$ and $\mu = 1.27$ with an error of no greater than 5%.

In a similar way, by carrying out direct measurements of the values of ψ and $S + R$ for different φ using the graph in Fig. 15 of [2] and the least-squares method, it is possible to obtain an approximation for ψ ($= \psi_{\text{isen}}$ in the notation of [2]) and $S + R$ ($= (\psi - \psi_{\text{isen}})$ in the notation of [2]). Then, by using the relation $R = 3S$, given in [2], we obtain

$$\psi(\varphi) = 4.5 - 13.5\varphi + 20.5\varphi^2 - 12.8\varphi^3 \tag{5.1}$$
$$S(\varphi) = 1.45 - 5.93\varphi + 8.04\varphi^2 - 3.44\varphi^3, \quad R(\varphi) = 4.36 - 17.8\varphi + 24.1\varphi^2 - 10.3\varphi^3$$

These formulae also have an error of about 5%, which was checked by observations of the results of varying the coefficients in these formulae. As the degree of the polynomials is increased, the accuracy with which their coefficients can be determined falls and when it is reduced the discrepancy increases.

In accordance with the recommendations in [2], the time lag coefficients for losses were taken as one and a half times greater than the average time needed for the fluid particle to pass through the channels between the blades. The transit time for each channel is equal to $l\cos\alpha/\varphi$ where l is the length of the chord and α is the stagger angle of the blade from Table 1 in the same paper. T he results were separately averaged over the stators and the rotors. We recall that the radius of the rotors, which, according to the same table, is equal to 305 mm, is used as the length scale. We therefore find

$$k_\sigma = 1.5 \times ((20+81)\cos 8.1° + 31\cos 11° + 31\cos 12° + 31\cos 5.5°)/(5 \times 305) \approx 1.5 \times 0.126 = 0.189$$
$$k_\tau = 1.5 \times (45\cos 42.8° + 45\cos 43.5° + 51\cos 44.6°)/(3 \times 305) \approx 1.5 \times 0.111 = 0.1665$$

Formulae (5.1) and the values of the constants were checked by calculating the eigenvalues of a problem on linear stability and by a comparison with the graphs in Fig. 20 of [2]. The resulting agreement confirmed that the parameters of the compressor are correctly approximated by the above formulae. Moreover, these formulae and the values of the parameters are also in agreement with formulae obtained directly from Haynes *et al.* through the intermediary of Dr M. Barnett (I wish to thank him for the information supplied), although, in [2], another (non-polynomial) approximation was used for ψ, S and R.

In the final section, by the term MIT compressor, we shall mean a compressor with the characteristics given above, that is, (5.1) and $k_\sigma = 0.189$, $k_\tau = 0.1665$, $\nu = 0.67$ and $\mu = 1.27$.

6. ANALYSIS OF AN ACTUAL COMPRESSOR

For the MIT compressor, a calculation using the linear theory gives $\varphi_c = 0.459$ and $V = 0.329$. The following values for the coefficients of Eq. (3.3) are obtained from (3.2): $a = -4.53 + 0.769i$; $b = -7.11 - 63.1i$.

If we ignore the non-linearity in the outlet channel of the compressor, i.e. we put $\pi_{k\lambda} = 0$ and $\pi_{jkl} = 0$, the first coefficient does not change and the value of the second coefficient only changes slightly: $b = -4.41 - 61.6i$. This means that, in the case of the MIT compressor, non-linearity in the outlet channel does not play any important role. This fact is used in the subsequent analysis, since the numerical calculations were carried out ignoring this non-linearity.

It was found that the real part of b is negative. This means that rotating stall is excited supercritically. However, it may also be pointed out that $|\mathrm{Re}\, b|$ is about one tenth of $|b|$. To answer the question as to what extent the result which has been obtained concerning the supercritical excitation of RS is stable to variations in the characteristics of the compressor (this is important since they can only be determined approximately), the function $\psi(\varphi)$ was slightly changed so as to obtain compressors with clearly defined super- and subcritical excitation while retaining the value of the critical flow rate and the phase velocity of a neutral perturbation. In the case of a supercritical compressor

$$\psi(\varphi) = 6.858 - 27.99\varphi + 50.05\varphi^2 - 32.80\varphi^3 \tag{6.1}$$

and, for a subcritical compressor

$$\psi(\varphi) = 2.142 + 0.987\varphi - 9.0505\varphi^2 + 7.200\varphi^3 \tag{6.2}$$

Although the coefficients of the polynomial expressions are quite different, the actual difference between the characteristics of these compressors is not large, as can be seen in Fig. 1, so that the characteristics introduced into the treatment, although they do not correspond to the MIT compressor, are still completely realistic samples of the characteristics of real compressors. Graphs of the function $\psi_r = \psi - S - R$ are shown in Fig. 1. The continuous curve corresponds to the MIT compressor (5.1) and the other curves correspond to compressors with a supercritical (dashed curve, (6.1)) and (dot-and-dash curve, (6.2)) excitation. When no account is taken of the non-linearity in the outlet channel, $a = -3.59 + 0.828i$, $b = -24.6-39.5i$ is obtained in the case of a supercritical compressor and $a = -5.48 + 0.709i$, $b = 19.4 - 88.7i$ in the case of a subcritical compressor.

The numerical experiments showed that RS in super- and subcritical compressors is excited super- and subcritically respectively, according to the predictions obtained from Landau's equation. However, in the case of the MIT compressor, the first numerical calculations revealed a subcritical nature of the excitation, contrary to the predictions

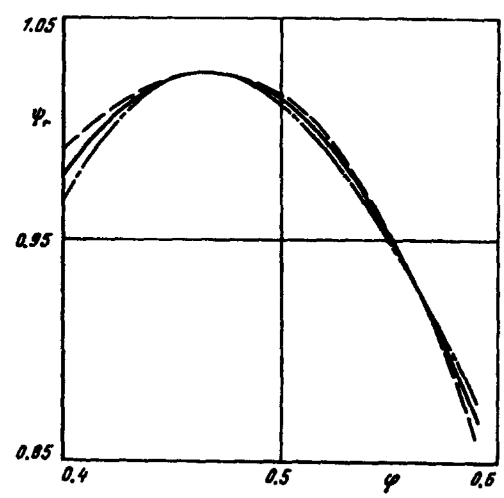

Fig. 1.

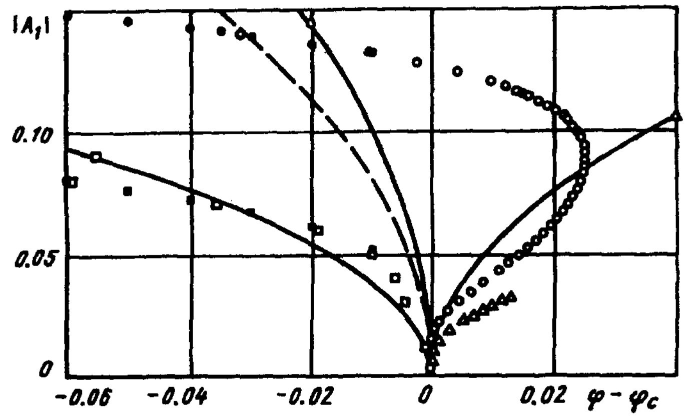

Fig. 2.

of the theory. This situation is illustrated in Fig. 2 where the different curves depict regimes which are steady with respect to their amplitude for the different compressors, i.e. obtained from Eqs (3.3) with $d|B_1|/dt = 0$ or numerically as described in Section 4. Moreover, the abscissa, which corresponds to zero amplitude, also depicts a regime with a steady amplitude. The dashed curve corresponds to the solution of Eq. (3.3) for the MIT compressor, with non-linearity in the outlet channel taken into account. All the remaining curves data ignore this non-linearity, that is, assume $\pi_{kl} = \pi_{jkl} = 0$. The continuous curves are the results of a calculation using Eqs (3.3); the light symbols are the results of direct numerical modelling, the dark symbols are the results of an approximate numerical calculation for a specified amplitude, the circles are the results for the MIT compressor, the squares are the results for a compressor with supercritical excitation (6.1) and the triangles are the results for a compressor with subcritical excitation (6.2).

In practice, only the corresponding stable regimes can be realized for each value of φ_s. The regime with a zero amplitude is stable when $\varphi_s > \varphi_c$. The stability of other regimes of the same compressor for the same φ_s can be obtained using a simple rule from the theory of bifurcations: stable and unstable branches will alternate when the amplitude is increased. For example, in the case of the MIT compressor with the characteristic obtained numerically (the small circles in Fig. 2), the intermediate branch is unstable and the branch with the greatest values of the amplitude is stable. Correspondingly, the behaviour of the compressor in the case of slow changes in φ_s, that is, in the flow rate through it, can be described in the following manner (if one ignores the details which are only obvious in Fig. 3). For sufficiently large φ_s, there is only one regime. There is no RS and this regime is stable. As soon as φ_s is reduced to values smaller than φ_c, this regime becomes unstable, and the amplitude increases and reaches a finite value corresponding to the upper branch of the amplitude curve and remains on this branch when the flow rate is further reduced. If, now, φ_s is increased to values greater than φ_c, the compressor will remain on the upper stable branch of the characteristic up to its disappearance, that is, up to a value of φ at which the graph of the amplitude has a vertical tangent. When φ_s is increased further, the RS disappears and the amplitude in the steady regime will be equal to zero. Hence, subcritical excitation of RS is accompanied by hysteresis. In the case of supercritical excitation, which corresponds to a supercritical compressor or the predictions of the weakly non-linear theory in the case of the MIT compressor, the amplitude changes continuously as $\sqrt{(\varphi_c - \varphi_s)}$ as φ_s passes through φ_c and there is no hysteresis.

A more detailed investigation showed that the predictions of the theory and the numerical calculations for the MIT compressor are, in fact, in complete agreement. However, the compressor performance turns out to be more complex. As $|A_1|$ is increased from zero, the curve for the dependence of the amplitude of RS on the flow rate in the MIT compressor (the small circles in Figs 2 and 3) initially deviates to the left, exactly in agreement with theory. It then deviates strongly to the right and only then again to the left. Here, the point is that Landau's equation is only applicable for small amplitudes and for small deviations of φ_s from φ_c. The parts of the graphs in Fig. 2 which are close to the critical point are shown on an enlarged scale in Fig. 3. These graphs demonstrate that the numerical results (the points) are in good agreement with the solution of Eqs (3.3) (the continuous curves) on approaching the critical point in the case of all three compressors. The MIT compressor is distinguished by the relative smallness of the second coefficient in Landau's equation. Hence, on increasing the amplitude of the perturbation, the non-linear terms of a higher order, which are ignored in the equation, lead not only to quantitative but also qualitative changes in the compressor performance. It can be seen from Figs 2 and 3 that, for the other compressors which have been considered also, the curves, as $|A_1|$ is increased, also deviate to the right from the predictions of the theory but not in such a pronounced manner as to change the sub- or supercritical nature of the excitation of stall.

Since a comparatively small change in the characteristic of the MIT compressor (see Fig. 1) can make the type of excitation of RS quite specific, it has to be concluded that the relative smallness of the second coefficient in Landau's equation for this compressor is an accidental circumstance which is not typical of the majority of other compressors.

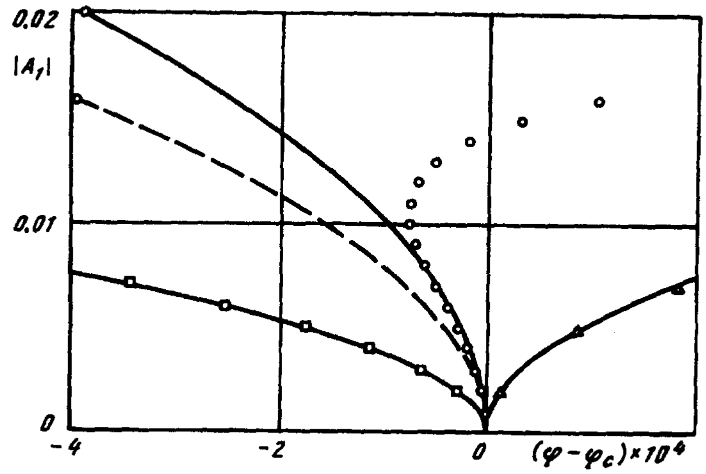

Fig. 3.

Note that it was found, in all of the cases considered, that Im $b < 0$. According to expression (3.4), this means that, as the amplitude is increased, the phase velocity of the wave increases.

7. CONCLUSION

The coefficients in Landau's equation for the amplitude of the first unstable mode of rotating stall (RS) can be expressed by explicit formulae in terms of the characteristics of a compressor which is described by a modified Moore–Greitzer model. The correctness of the formulae is confirmed by numerical calculations. The formulae obtained enable one to predict whether the excitation of RS will be sub- or supercritical, that is, whether there is hysteresis in the RS when there are slow (quasi-steady) changes in the flow rate through the compressor. The application of the formulae obtained to the laboratory compressor used at the Massachusetts Institute of Technology for studying RS showed that this compressor is atypical in the sense that it occupies an intermediate position between the two cases which have been indicated: a comparatively small change in the characteristics can convert it into a compressor with sub- or supercritical RS excitation.

It has not been ruled out that this special feature leads to a wider than usual range of amplitude of the perturbations in which linear theory turns out to be adequate, which is possibly an advantage in attaining active control of the compressor. Furthermore, in this compressor the weakness of the non-linear effects could be responsible for the absence of the alternative scenario [6] of a transition to RS associated with the development of a localized perturbation.

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